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ON FIXING ELEMENTS IN MATROID MINORS

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Let **F** be a collection of 3-connected matroids which is (3, 1)-rounded, that is, whenever a 3-connected matroid M has a minor in **F** and e is an element of M, then M has a minor in **F** whose ground set contains. e. The aim of this note is to prove that, for all sufficiently large n, the collection of n-element 3-connected matroids having some minor in **F** is also (3, 1)-rounded.

1. Introduction

This paper is concerned with relating certain minors that are known to occur in a matroid to particular elements of the matroid. Most of the matroid terminology used here will follow Welsh [17]. If T is a subset of the ground set $E(M)$ of a matroid M, we shall say that M *uses* T. The deletion and contraction of T from M will be denoted by $M\setminus T$ and M/T respectively. Flats of M of ranks one and two will be called *points* and *lines*. The $(r+1)$-vertex wheel, the whirl of rank r [17, pp. 80—81], the Fano matroid and the uniform matroid of rank k on an n-element set will be denoted by W_r, W^r, F_7 and $U_{k,n}$ respectively. If M' is a matroid such that $E(M')\cap E(M)\supseteq\{e\}$, then M' is *e-isomorphic* to M if there is an isomorphism between M' and M under which e is fixed.

If M_1 and M_2 are matroids on the sets S and $S\cup e$ where $e\notin S$, then M_2 is an *extension* of M_1 if $M_2\setminus e=M_1$. Let F be a flat of M_1. Then M_1 has a unique extension M_2 on $E(M_1)\cup e$ such that the flats of M_2 in which e occurs other than as a coloop are precisely the sets of the form $F'\cup e$ for which F' is a flat of M_1 containing F [5]. This extension is said to be obtained from M_1 by adding *e freely* to F or, if $F=E(M_1)$, by adding e freely to M_1.

For a positive integer k, the matroid M is *k-connected* [16] if, for all $j<k$, there is no partition $\{X, Y\}$ of $E(M)$ such that $|X|, |Y|\geqq j$ and $r(X)+r(Y)-r(M)\leqq j-1$.

Let **F** be a class of matroids. The matroid N' is an **F**-*minor* of the matroid N if N' is a minor of N isomorphic to some member of **F**. For k and l positive integers, **F** is (k, l)-*rounded* if every member of **F** is k-connected and the following condition holds.

If M is a k-connected matroid having an **F**-*minor and X is a subset of $E(M)$ with at most l elements, then M has an* **F**-*minor using X.*

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This definition follows Bixby and Coullard [3] except that they also require that every member of **F** has at least four elements. It extends an earlier definition of Seymour [12] who defined a class of matroids to be "l-rounded" if it is $(l+1, l)$-rounded in the above sense. The initial motivation for these definitions derived from a sequence of results for non-binary matroids [1, 11, 6, 9]. The following result of Seymour [10, 11] considerably simplifies the task of testing whether a collection of matroids is $(l+1, l)$-rounded when l is 1 or 2.

Theorem 1.1. *Let l be 1 or 2 and* **F** *be a collection of $(l+1)$-connected matroids. Then* **F** *is $(l+1, l)$-rounded if and only if the following condition holds.*

If M is an $(l+1)$-connected matroid having a single-element deletion or a single-element contraction which is an **F***-minor and X is a subset of $E(M)$ with at most l elements, then M has an* **F***-minor using X.*

Relatively little attention has been devoted to identifying large numbers of examples of collections of matroids with the properties defined above. The next result uses a given (3, 1)-rounded collection to construct an infinite sequence of disjoint (3, 1)-rounded collections.

Theorem 1.2. *Let* **F** *be a collection of 3-connected matroids and* $\mathbf{F}_n$ *be the class of n-element 3-connected matroids having some minor isomorphic to a member of* **F**. *If* **F** *is (3, 1)-rounded, then so is* $\mathbf{F}_n$ *provided that either*

(i) **F** *has finitely many non-isomorphic members and*

$$n \geqq \max\{|E(F)|:\ F \in \mathbf{F}\}$$

or

(ii) **F** *does not contain two members F_1 and F_2 such that F_2 is isomorphic to a minor of F_1 and*

$$|E(F_1)|-|E(F_2)| = 1.$$

This theorem remains true if "3" is replaced by "2" throughout. This is an easy consequence of the following result of Seymour [10, p. 290].

Lemma 1.3. *If N is a 2-connected minor of a 2-connected matroid M and $x \in E(M)-E(N)$, then at least one of $M \setminus x$ and M/x is 2-connected and has N as a minor.* ▮

The generalization of this lemma to 3-connected matroids is not true. The following result of Coullard (private communication) forms the basis of the proof of Theorem 1.2.

Theorem 1.4. *Let N be a 3-connected proper minor of a 3-connected matroid M and suppose that $e \in E(N)$. Then, provided M is neither a wheel nor a whirl of rank at least three, there is an element f of M such that, for M_0 equal to one of $M \setminus f$ and M/f, M_0 is 3-connected and has a minor N' that is e-isomorphic to N.*

If $|E(N)| \geqq 4$, this result is an immediate consequence of a theorem of Tseng and Truemper [13, Theorem 2.2]. For $|E(N)| \leqq 3$, our proof differs from Coullard's and will be given in the next section.

2. The proofs

In this section we shall prove Theorems 1.2 and 1.4 and discuss some applications of the former.

Proof of Theorem 1.4. By Bixby and Coullard's extension [12, Theorem 4.4] of a result of Truemper [14], there is a proper 3-connected minor M' of M such that N is a minor of M' and $|E(M)-E(M')|\leqq 3$. It suffices to prove Theorem 1.4 in the case that $M'=N$. If $|E(M')|\geqq 4$, then, as noted in the introduction, the theorem follows from a result of Tseng and Truemper [13, Theorem 2.2]. Thus we may assume that $|E(M')|\leqq 3$. Then M' is uniform. Since the automorphism group of an m-element uniform matroid is the symmetric group S_m, if M is also uniform, then the theorem follows. Furthermore, if M is not uniform, it suffices to show that M has a single-element deletion or contraction that uses e and is uniform of rank and corank at least two.

Now $|E(M')|\leqq 3$ and $|E(M)-E(M')|\leqq 3$, so $|E(M)|\leqq 6$. Moreover, since we may assume that M is not uniform and M is neither a wheel nor a whirl, it follows, by [8, Theorem 2.5], that M is isomorphic to P_6 or Q_6, where Euclidean representations for the last two matroids are shown in the figure. In each case, with the elements labelled as there, $M\setminus p\cong U_{3,5}$ and $M/q\cong U_{2,5}$, so the required uniform single-element deletion or contraction of M using e exists. ∎

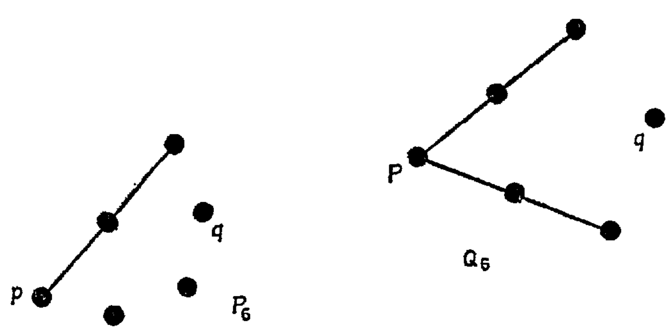

Fig. 1

Proof of Theorem 1.2. Since every member of $\mathbf{F}_n$ is 3-connected, it suffices to show that $\mathbf{F}_n$ is (2, 1)-rounded. Now suppose M is a 2-connected matroid having an $\mathbf{F}_n$-minor M_1 where $E(M)-E(M_1)=\{e\}$. We may assume, by duality, that $M_1=M\setminus e$. If M is not 3-connected, then e is parallel to some element f of M [7, Lemma 2.1] and so $M\setminus f$ is an $\mathbf{F}_n$-minor of M_1 using e. Thus we may suppose that M is 3-connected.

Suppose that $M\setminus e\in\mathbf{F}$. Then, by assumption, $M\notin\mathbf{F}$ and, since $\mathbf{F}$ is (3, 1)-rounded, M has a proper $\mathbf{F}$-minor N using e. Applying Theorem 1.4 to M and N gives the result.

Assume that $M\setminus e\notin\mathbf{F}$. Then $M\setminus e$ has a proper minor N_1 in $\mathbf{F}$. By Lemma 1.3, M has a connected minor N_2 such that N_1 is $N_2\setminus e$ or N_2/e. As N_1 is 3-connected, N_2 is 3-connected unless e is in series or parallel with some element f of N_1 [7, Lemma 2.1]. In the exceptional cases, let N_3 equal N_2/f or $N_2\setminus f$, respectively. If N_2 is 3-connected, let N_3 be an $\mathbf{F}$-minor of N_2 using e, the existence of which is guaranteed by the fact that $\mathbf{F}$ is (3, 1)-rounded. To complete the proof, we now apply Theorem 1.4 letting $N=N_3$. ▮

The rest of this section looks at various consequences of Theorem 1.2.

Corollary 2.1. *For all positive integers n, each of the following classes is* (3, 1)*-rounded:*
(i) *n-element* 3*-connected non-binary matroids;*
(ii) *n-element* 3*-connected non-ternary matroids;*
(iii) *n-element* 3*-connected non-regular matroids.*

Proof. By [1] and [10], the collections of minor-minimal matroids which are non-binary, non-ternary, and non-regular are (3, 1)-rounded. ▮

In the next result, the restriction on n in (i) of Theorem 1.2 takes effect.

Corollary 2.2. *The class of n-element* 3*-connected non-graphic matroids is* (3, 1)*-rounded for all* $n\geqq 10$.

Proof. By [10, p. 29], the set

$$\mathbf{F} = \{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K_{3,3}+e)\}$$

is (3, 1)-rounded, where $K_{3,3}+e$ is the graph obtained from $K_{3,3}$ by adding an edge joining two non-adjacent vertices. Moreover, every non-graphic matroid has a minor isomorphic to a member of the set

$$\mathbf{E} = \{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})\} \quad [15].$$

The required result now follows using the fact that, for all $n\geqq 10$, $\mathbf{F}_n=\mathbf{E}_n$. ▮

3. Some counterexamples

In this section we give counterexamples to some possible strengthenings and extensions of Theorems 1.2 and 1.4.

First we observe that if $\mathbf{F}$ does not satisfy (i) or (ii) of Theorem 1.2, then the theorem may fail. To see this, let

$$\mathbf{F} = \{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K_{3,3}+e)\}.$$

Then, as noted in the proof of Corollary 2.2, $\mathbf{F}$ is (3, 1)-rounded. Clearly $\mathbf{F}$ fails to satisfy (ii) of Theorem 1.2. Now consider $\mathbf{F}_9$ noting that $9<\max\{|E(F)|: F\in\mathbf{F}\}$. The matroid $M^*(K_{3,3}+e)$ is certainly 3-connected and has an $\mathbf{F}_9$-minor, namely $M^*(K_{3,3})$. Moreover, as none of $U_{2,4}$, F_7 or F_7^* is cographic, $M^*(K_{3,3})$ is the only cographic member of $\mathbf{F}_9$. As $M^*(K_{3,3}+e)$ is cographic, if it has an $\mathbf{F}_9$-minor containing e, then this minor must be isomorphic to $M^*(K_{3,3})$. It is easy to check that no such minor exists and we conclude that $\mathbf{F}_9$ is not (3, 1)-rounded.

A second direction in which one may attempt to extend Theorem 1.2 is to cover the case when $\mathbf{F}$ is (3, 2)-rounded. Will this assumption guarantee that $\mathbf{F}_n$ is (3, 2)-rounded? To answer this, we take the (3, 2)-rounded collection $\mathbf{F}=\{U_{2,4}\}$. Then

$$\mathbf{F}_4 = \mathbf{F},$$

$$\mathbf{F}_5 = \{U_{2,5}, U_{3,5}\}$$

and

$$\mathbf{F}_6 = \{U_{2,6}, U_{4,6}, U_{3,6}, W^3, P_6, Q_6\}.$$

Evidently $\mathbf{F}_4$ is (3, 2)-rounded. Furthermore, $\mathbf{F}_6$ is (3, 2)-rounded. This follows easily from the fact that if M is a 3-connected non-binary matroid of rank and corank at least 3 and $\{x, y\} \subseteq E(M)$, then M has a minor isomorphic to one of $U_{3,6}$, W^3, P_6 or Q_6 that uses $\{x, y\}$ [8, Corollary 3.5].

Although both $\mathbf{F}_4$ and $\mathbf{F}_6$ are (3, 2)-rounded, $\mathbf{F}_n$ is not (3, 2)-rounded for any other n exceeding 3. To see that $\mathbf{F}_5$ is not (3, 2)-rounded, we observe that Q_6 has an $\mathbf{F}_5$-minor but has no such minor using $\{p, q\}$. Indeed, we can generalize this example to show that, for all $m \geqq 2$, $\mathbf{F}_{2m+1}$ is not (3, 2)-rounded. Let U_m be the matroid obtained by taking the parallel connection of m 3-point lines [4] and adding a point q freely to this matroid, that is, $U_m \setminus q$ has rank $m+1$ and consists of m distinct 3-point lines all passing through a common point, say p. It is straightforward to check that both $U_m \setminus p$ and U_m / q are $\mathbf{F}_{2m+1}$-minors of U_m but that U_m has no $\mathbf{F}_{2m+1}$-minor using $\{p, q\}$.

To see that, for all $m \geqq 4$, $\mathbf{F}_{2m}$ is not (3, 2)-rounded, consider the matroid V_m that is formed from $M(W_m)$ by adding the element e freely on the line $\{a_1, a_3\}$, where the spokes of $M(W_m)$ are labelled, in cyclic order, $a_1, a_2, a_3, \ldots, a_m$. It is not difficult to check that both $V_m \setminus a_1$ and $V_m \setminus a_3$ are $\mathbf{F}_{2m}$-minors of V_m, but that V_m has no $\mathbf{F}_{2m}$-minor using $\{a_1, a_3\}$.

We can use one of the above examples to show that Theorem 1.4 cannot be strengthened by requiring M_0 to contain an N-minor using two nominated elements of N. In particular, if $M = U_3$ and $N = Q_6$, then M has N as a minor. However, no single-element deletion or contraction of M has an N-minor using both p and q.

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